

# Any Finite Group is the Group of Some Binary, Convex Polytope

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**Abstract** For any given finite group, Schulte and Williams (2015) produce a convex polytope whose combinatorial automorphisms form a group isomorphic to the given group. We provide here a shorter proof for a stronger result: the convex polytope we build for the given finite group is binary, and even combinatorial in the sense of Naddef and Pulleyblank (1981); the diameter of its skeleton is at most 2; any automorphism of the skeleton is a combinatorial automorphism; any combinatorial automorphism of the polytope is induced by some isometry of the space.

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## 1 Introduction

A *combinatorial automorphism* of a convex polytope is a permutation of its set of vertices that maps the set of vertices of any face to the set of vertices of some face. Recently, Schulte & Williams (2015) established that any finite group is isomorphic to the group of combinatorial automorphisms of some convex polytope. As their proof is rather long and involved, we propose here a shorter one which even establishes a stronger result, because the convex polytope we build for a given group is always *binary*: all its vertices have coordinates equal to 0 or 1. The polytope is moreover *combinatorial* in the sense of Naddef & Pulleyblank (1981): it is a binary polytope on which every two nonadjacent vertices have their midpoint equal to the midpoint of two other vertices (such combinatorial polytopes appear also, for instance, in Matsui & Tamura, 1995).

**Theorem 1.1.** *For each finite group  $G$  there exist a natural number  $d$  and a convex polytope  $P_G$  in  $\mathbb{R}^d$  satisfying Properties (i)–(iv):*

- (i) *the group of combinatorial automorphisms of the convex polytope  $P_G$  is isomorphic to  $G$ ;*
- (ii) *any automorphism of the skeleton (graph) of  $P_G$  is a combinatorial automorphism of  $P_G$ ;*
- (iii) *any combinatorial automorphism of  $P_G$  is the restriction to the set of vertices of some isometry of  $\mathbb{R}^d$  stabilizing  $P_G$ ;*
- (iv) *the convex polytope  $P_G$  is binary, and even combinatorial;*

(v) *the diameter of the skeleton of  $P_G$  is at most 2.*

In all the paper, we denote by  $G$  a given, finite group. The required polytope  $P_G$  for  $G$  arises from a two-step construction. In Section 2 we recall results on the existence of a graph  $\Gamma = \Gamma(G)$  whose automorphism group is (isomorphic to)  $G$ . In Section 3 we build for almost any graph  $\Gamma$  a polytope  $P = P(\Gamma)$  whose combinatorial automorphism group is isomorphic to the automorphism group of the graph  $\Gamma$ . It happens that  $P(\Gamma)$  is a combinatorial polytope; moreover, any combinatorial automorphism of the polytope  $P(\Gamma)$  is induced by some isometry of the space (Proposition 3.1). In passing, we notice that any graph on  $d$  nodes is an induced subgraph of the graph of some binary, combinatorial, convex polytope of dimension  $d$ .

For any given group  $G$ , the two-step construction produces a binary, combinatorial, convex polytope  $P_G = P(\Gamma(G))$  which satisfies the properties in Theorem 1.1. In Section 4, after completing the proof of the theorem, we discuss how to decrease the dimension and/or the number of vertices of the polytope  $P_G$ . We also mention a curious, immediate consequence: for any finite group  $G$ , there is a directed graph for which the asymmetric travelling-salesman polytope has its automorphism group isomorphic to  $G$ .

Let us mention that Babai (1977) characterizes the isometry groups of convex polytopes (binary or not) which are transitive on the set of vertices. The only excluded groups are the generalized dicyclic groups and the abelian groups of exponent at least 2 (see Babai, 1977, or Babai & Godsil, 1982, for a definition of generalized dicyclic groups and, for instance, Robinson, 1996, for further group terminology).

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## 2 Building a Graph for any Group

Our graphs  $\Gamma = (V, E)$  have neither loops nor multiple edges. We use the terms ‘node’ and ‘link’ for the elements of respectively  $V$  and  $E$  (while keeping ‘vertex’ and ‘edge’ for polytopes). Let again  $G$  be a finite group. Frucht (1939) was the first to show the existence of some graph whose automorphism group is isomorphic to  $G$  (for a historical perspective, see Hevia, 1995). For reasons which will become clear in Section 4, we would like to select, given the group  $G$ , a graph having  $G$  as its automorphism group and moreover having the least possible number of nodes.

Babai (1974) proves that, with the exception of the cyclic groups of orders 3, 4 and 5, there exists, for any finite group  $G$ , a graph having at most  $2|G|$  nodes with automorphism group  $G$ . For many finite groups  $G$ , there is a graph on  $|G|$  nodes or less with the stronger property that its automorphism group is isomorphic to  $G$  and acts regularly on the set of nodes. As found by Godsil (1978) and reported by Babai & Godsil (1982), such a graph exists for any finite group  $G$  except for

- the abelian groups of exponent at least 3;

- the generalized dicyclic groups;
- 13 other groups of orders at most 32.

Now forgetting about regularity, we mention that Arlinghaus (1985) provides for each finite, abelian group  $G$  the minimum number of nodes in a graph having  $G$  as its automorphism group.

### 3 Building a Binary Polytope for any Graph

Let  $\Gamma = (V, E)$  be a graph. We associate to any subset  $S$  of  $V$  its *characteristic vector*  $\chi(S)$  in  $\mathbb{R}^V$ , defined by

$$\chi(S)_v = \begin{cases} 1 & \text{if } v \in S, \\ 0 & \text{if } v \notin S. \end{cases}$$

The *polytope*  $P(\Gamma)$  of the graph  $\Gamma = (V, E)$  is the convex hull in  $\mathbb{R}^V$  of the characteristic vectors of the empty set, the one-element subsets of  $V$  and the links in  $E$ :

$$P(\Gamma) = \text{conv}(\{\chi(\emptyset)\} \cup \{\chi(\{v\}) \mid v \in V\} \cup \{\chi(e) \mid e \in E\}). \quad (1)$$

In the next proposition, we consider isometries of the euclidean vector space  $\mathbb{R}^V$  whose dot product makes the canonical basis of  $\mathbb{R}^V$  an orthonormal basis.

**Proposition 3.1.** *For any graph  $\Gamma = (V, E)$ , Assertions (a)–(d) hold:*

- (a) *the convex polytope  $P(\Gamma)$  has dimension  $|V|$ ;*
- (b)  *$P(\Gamma)$  is binary, and even combinatorial;*
- (c) *the diameter of the skeleton of  $P(\Gamma)$  is at most 2;*
- (d) *the group of combinatorial automorphisms of  $P(\Gamma)$  is isomorphic to the automorphism group of the graph  $\Gamma$  if and only if*

[C] *the graph has at least one link ( $E \neq \emptyset$ ) and there does not exist any bipartition of  $V$  into two stable sets  $A, B$  with some node in  $A$  adjacent to all nodes in  $B$ ;*

- (e) *when the latter Condition [C] holds, each automorphism of the graph which is the skeleton of the polytope  $P(\Gamma)$  is a combinatorial automorphism of the polytope  $P(\Gamma)$ ;*
- (f) *when the latter Condition [C] holds, each combinatorial automorphism of  $P(\Gamma)$  is induced by some isometry of  $\mathbb{R}^V$  stabilizing  $P(\Gamma)$ .*

*Proof.* Because characteristic vectors have all their coordinates equal to 0 or 1, the vertices of the polytope  $P(\Gamma)$  are all the characteristic vectors appearing in Equation (1). Assertion (a) follows. We next describe the adjacency between vertices of the polytope  $P(\Gamma)$ :

- (i)  $\chi(\emptyset)$  is adjacent to each  $\chi(\{v\})$  (for  $v$  in  $V$ ), and to no other vertex (because adjacency of the same vertices hold on the unit cube);
- (ii) the vertices  $\chi(\{u\})$  and  $\chi(\{v\})$ , for distinct nodes  $u, v$  in  $V$ , are adjacent if and only if  $u$  and  $v$  are not linked in  $\Gamma$ . Indeed, if  $u$  and  $v$  are linked, then the vertices  $\chi(\emptyset)$  and  $\chi(\{u, v\})$  have the same midpoint as the vertices  $\chi(\{u\})$  and  $\chi(\{v\})$  do. If  $u$  and  $v$  are not linked, then the affine inequality on  $R^V$

$$x_u + x_v - 2 \sum_{i \in V \setminus \{u, v\}} x_i \leq 1$$

defines a face of  $P(\Gamma)$  which is the segment  $[\chi(\{u\}), \chi(\{v\})]$  (meaning the inequality is valid for  $P(\Gamma)$  and it is satisfied with equality only at the points of  $P(\Gamma)$  which belong to the segment).

- (iii) two vertices  $\chi(\{v\})$  and  $\chi(\{u, w\})$ , for  $v \in V$  and  $\{u, w\} \in E$ , are adjacent if and only if either  $v \in \{u, w\}$  or ( $v \notin \{u, w\}$  and  $v$  is linked to neither  $u$  nor  $w$ ). If  $v \in \{u, w\}$ , adjacency in  $P(\Gamma)$  results from adjacency in the unit cube. If  $v \notin \{u, w\}$ , notice that if  $\{u, v\} \in E$ , then  $\chi(\{v\})$  and  $\chi(\{u, w\})$  have the same midpoint as  $\chi(\{w\})$  and  $\chi(\{u, v\})$ , so  $\chi(\{v\})$  and  $\chi(\{u, w\})$  are not adjacent. The argument is similar if  $\{v, w\} \in E$ . Conversely, if  $v \notin \{u, w\}$  and  $\{u, v\}, \{v, w\} \notin E$ , the affine inequality

$$x_v + \frac{1}{2}x_u + \frac{1}{2}x_w - 2 \sum_{i \in V \setminus \{u, v, w\}} x_i \leq 1$$

defines a face of  $P(\Gamma)$  which is  $[\chi(\{v\}), \chi(\{u, w\})]$ .

- (iv) for distinct links  $e, f$  in  $E$ , the vertices  $\chi(e)$  and  $\chi(f)$  are adjacent if and only if

either  $e$  and  $f$  have a common node, or  $e$  and  $f$  are disjoint and not contained in any four-cycle in the graph  $G$ .

The last criterion (which can be proved with arguments similar to those used in (i)–(iii)) is valid also for adjacency of vertices of the *edge polytope*  $P_E = \text{conv}(\{\chi(e) \mid e \in E\})$  of the graph  $G$  (see for instance Tran & Ziegler, 2014). It also characterizes adjacency of vertices of  $P(\Gamma)$  because  $P_E$  is the face of  $P(\Gamma)$  defined by the inequality  $\sum_{i \in V} x_i \leq 2$ .

Assertions (b) and (c) in the statement follow from the description we just gave of the adjacency of vertices. Moreover, the vertex  $\chi(\emptyset)$  has the following property:

- (\*) the graph induced on the neighborhood of the vertex in the skeleton of  $P(\Gamma)$  is isomorphic to the complement of the graph  $\Gamma$ .

We show that if the graph  $\Gamma$  has at least one link and there is some vertex of the polytope  $P(\Gamma)$ , distinct from  $\chi(\emptyset)$ , which also has Property (\*), then there exists a bipartition of  $V$  as in Condition [C].

If some vertex  $\chi(\{v\})$ , for  $v \in V$ , satisfies Property (\*), its number of neighbours must be  $|V|$ . Because  $\chi(\{v\})$  is for sure adjacent to  $\chi(\emptyset)$ , to  $\chi(\{u\})$  when  $v \not\sim u$  and to  $\chi(\{v, w\})$  when  $v \sim w$ , this implies that  $\chi(\{v\})$  cannot be adjacent to any further vertex. In particular, there is no link between any two nodes of  $\Gamma$  not linked to  $v$ . Then the neighborhood of  $\chi(\{v\})$  is the union of two cliques, respectively the clique  $\mathcal{A}$  formed by the vertex  $\chi(\emptyset)$  together with the vertices  $\chi(\{u\})$  for  $v \not\sim u$ , and the clique  $\mathcal{B}$  formed by the vertices  $\chi(\{v, w\})$  for  $v \sim w$ . Moreover,  $\chi(\emptyset)$  is adjacent to no vertex in  $\mathcal{B}$ . Thus, by Property (\*), there exists a bipartition  $A, B$  as in Condition [C].

If some vertex  $\chi(\{v, w\})$ , for  $\{v, w\} \in E$ , satisfies Property (\*), it has  $|V|$  neighbours. As  $\chi(\{v, w\})$  is adjacent to  $\chi(\{v\})$ , to  $\chi(\{w\})$ , to all  $\chi(\{v, u\})$  for  $v \sim u$ , to all  $\chi(\{w, u\})$  for  $w \sim u$  and to all  $\chi(\{u\})$  for  $v \not\sim u$  and  $w \not\sim u$ , we see that the neighborhoods  $N(v)$  of  $v$  and  $N(w)$  of  $w$  must be disjoint and that the nodes linked to neither  $v$  nor  $w$  must form a stable set; also, a node in  $\{v\} \cup N(v)$  is never linked to a node linked to neither  $v$  nor  $w$ . Then the neighborhood of  $\chi(\{v, w\})$  is the union of two cliques, respectively the clique  $\mathcal{A}$  formed by the vertex  $\chi(\{v\})$ , the vertices  $\chi(\{v, u\})$  for  $u \in N(v)$  and the vertices  $\chi(\{u\})$  for  $u$  linked to neither  $v$  nor  $w$ , and the clique  $\mathcal{B}$  formed by the vertices  $\chi(\{w\})$  and  $\chi(\{w, u\})$  for  $u \in N(w)$ . Moreover,  $\chi(\{v\})$  is adjacent to no vertex in  $\mathcal{B}$ . By Property (\*) there exists a bipartition  $A, B$  of  $V$  as in Condition [C].

We now prove Assertions (d) and (e) at the same time. Any automorphism of the graph  $\Gamma$  induces a linear permutation of  $\mathbb{R}^V$  which permutes the basis vectors. The linear permutation stabilizes the polytope  $P(\Gamma)$  (it permutes the vertices of  $P(\Gamma)$  as  $\chi(\{v, w\}) = \chi(\{v\}) + \chi(\{w\})$ ) and so it induces a combinatorial automorphism of the polytope, which is of course an automorphism of the skeleton. We derive an injective morphism of the group  $\text{Aut}(\Gamma)$  into the automorphism group of the skeleton. If Condition [C] holds the morphism is also surjective. Indeed, we just proved that  $\chi(\emptyset)$  is then the only vertex of  $P(\Gamma)$  which satisfies Property (\*). We conclude that the neighborhood of  $\chi(\emptyset)$  in the polytope skeleton is stabilized by any combinatorial automorphism of the polytope  $P(\Gamma)$ . We saw moreover that the graph induced on the neighborhood of  $\chi(\emptyset)$  is the complement of the graph  $\Gamma$ . So there cannot be more automorphisms of the skeleton of  $P(\Gamma)$  than of  $\Gamma$ . Next, to prove the necessity of Condition [C], assume there exist some bipartition of  $V$  into stable subsets  $A, B$  and some node  $v$  in  $A$  which is adjacent to all nodes of  $B$ . Consider the affine permutation  $\alpha$  mapping the point  $x$  of  $\mathbb{R}^V$  to the point  $x'$  of  $\mathbb{R}^V$  with

$$x'_u = \begin{cases} 1 - \sum_{a \in A} x_a & \text{if } u = v, \\ x_u & \text{if } u \neq v. \end{cases}$$

It is easily checked that  $\alpha$  permutes the vertices of the polytope  $P(\Gamma)$ , more precisely:  $\alpha$  exchanges  $\chi(\emptyset)$  and  $\chi(\{v\})$  and, for any node  $b$  in  $B$ , it exchanges  $\chi(\{b\})$  with  $\chi(\{v, b\})$ ; also,  $\alpha$  fixes all other vertices of  $P(\Gamma)$  (even those coming from further potential links between  $A$  and  $B$ ).

Finally, we prove Assertion (f). Let  $\beta$  be any combinatorial automorphism of  $P(\Gamma)$ . Then, as we saw,  $\beta$  fixes  $\chi(\emptyset)$  (the origin of  $\mathbb{R}^V$ ) and permutes among themselves the vectors  $\chi(u)$ , for  $u \in V$  (the basis vectors). There is an isometry  $\gamma$  of  $\mathbb{R}^V$  which fixes the origin and acts on the vectors  $\chi(u)$  exactly as  $\beta$  does. Any remaining vertex of  $P(\Gamma)$  has the form  $\chi(\{u, v\})$  (for some  $\{u, v\}$  in  $E$ ) and is thus equal to  $\chi(\{u\}) + \chi(\{v\})$ ; moreover, it is with  $\chi(\emptyset)$  the only further vertex in the smallest face of  $P(\Gamma)$  containing  $\chi(\{u\})$  and  $\chi(\{v\})$ . Hence, the restriction of the isometry  $\gamma$  to the set of vertices of  $P(\Gamma)$  equals  $\beta$ .  $\square$

From Proposition 3.1, we derive the following: for any graph  $\Gamma$  on  $m$  nodes with  $m \geq 3$ , there is a graph  $\Delta$  on at most  $\binom{m+1}{2}$  vertices which is  $m$ -connected, has diameter at most 2 and whose group of automorphisms is isomorphic to that of  $\Gamma$  (all finite groups appear in this way). Indeed, if  $\Gamma$  (resp. the complement  $\bar{\Gamma}$  of  $\Gamma$ ) satisfies Condition [C], we may take for  $\Delta$  the skeleton of the polytope  $P(\Gamma)$  (resp.  $P(\bar{\Gamma})$ ); the only five  $(2 + 2 + 1)$  remaining graphs  $\Gamma$  are easily handled.

## 4 Building a Binary Polytope for any Group

Given any finite group  $G$ , we now build a convex polytope  $P_G$  as in Theorem 1.1. As recalled in Section 2, there exists a graph  $\Gamma = \Gamma(G)$  whose automorphism group is  $G$ . If  $\Gamma$  does not satisfy Condition [C] of Proposition 3.1, we replace the graph  $\Gamma$  with its complement but keep the notation  $\Gamma$  for the resulting graph. Notice that now the graph  $\Gamma$  satisfies Condition [C], except for the graph having only one node, and for three particular pairs of complementary graphs on 2, 3 or 4 nodes respectively. In the first case, we can take for our convex polytope  $P_G$  a one vertex polytope; in each of the three particular cases, the automorphism group of the graph(s) is the cyclic group  $C_2$  and we can take for our polytope  $P_G$  the segment in  $\mathbb{R}^1$  with endpoints 0 and 1. Thus if the group  $G$  has at least three elements, we may apply Proposition 3.1: the resulting polytope  $P_G = P(\Gamma(G))$  satisfies all six properties in Theorem 1.1.

Let us discuss how large our polytope  $P_G = P(\Gamma(G))$  is with respect to the order  $n$  of the group  $G$ . First, recall that for any graph  $\Gamma$  the dimension of  $P(\Gamma)$  equals the number of nodes in  $\Gamma$ . We saw in Section 2 that there is a graph  $\Gamma(G)$  with at most  $2n$  nodes, except for the three exceptional groups  $C_3$ ,  $C_4$  and  $C_5$ . So apart from three exceptional groups, a careful choice of the graph  $\Gamma$  delivers a polytope  $P_G = P(\Gamma(G))$  of dimension at most  $2n$ . For  $C_3$ , there is a graph with 9 nodes, for  $C_4$  with 10 nodes, for  $C_5$  with 15 nodes such that all three graphs satisfy Condition [C]. It might be that another construction delivers polytopes with the desired properties and dimension less than 9, 10 and 15 respectively. Furthermore, in view of the result of Babai & Godsil (1982) recalled in Section 2, we can even build the polytope  $P_G$  in a space of dimension at most  $n$  when  $G$  does not belong to the list of exceptional groups given in Section 2.

What about the number of vertices of  $P_G = P(\Gamma(G))$ ? Assuming that the dimension of the polytope has been minimized, we might want to decrease the number of vertices as much as possible. In other words, having minimized the number of nodes in the graph  $\Gamma$  with automorphism group  $G$ , we might want to minimize the number of links. Characterizations of such minimal graphs exist for certain groups, see for instance McCarthy & Quintas (1979). If we agree to increase the dimension of the polytope  $P_G$  to (at most)  $2n + 1$ , where again  $n = |G|$ , we may use a result of Babai (1981): if  $g$  is the minimum number of generators of  $G$  and  $G$  is neither  $C_3$ ,  $C_4$  or  $C_5$ , there is a graph  $\Gamma$  with automorphism group  $G$  and at most  $2n + 1$  nodes and  $2n(g + 1)$  links. Our construction then produces a polytope of dimension at most  $2n + 1$  and with at most  $2n(g + 2) + 2$  vertices.

As announced in the introduction, we deduce that for any finite group  $G$ , there is a directed graph whose asymmetric traveling-salesman polytope has its automorphism group isomorphic to  $G$ . This follows at once from Theorem 1.1 combined with a nice result of Billera & Sarangarajan (1996), which states that any binary, convex polytope is isomorphic to the asymmetric traveling-salesman polytope of some directed graph.

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